

AREAL OPTIMIZATION OF POLYGONS¹

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Abstract

We will first solve the following problem analytically: given a piece of wire of specified length, we will find where the wire should be cut and bent to form two regular polygons not necessarily having the same number of sides, so that the combined area of the polygons thus formed is maximized, minimized, greater than, and less than a specified area. We will extend the results to the cases where the wire is divided into three and finally into an arbitrary number of segments. The second problem we will solve is as follows: two wires of specified length are to be bent into two regular polygons whose total number of sides is fixed. We will determine how the total number of polygonal sides are to be allocated between the wires so that the total area of the polygons is maximized. We will extend the results found here to the case where we are given any number of wires of specified length.

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1 Introduction

A segment of length, x , is to be cut from a wire of length, L . This segment, along with the remaining segment of length $L - x$, are then bent to form two regular polygons not necessarily having the same number of sides. We will begin by finding the total area, A_{total} , of these two polygons in terms of x and L . From this general form, we will solve the extrema problem; that is, we will compute x — the place where the wire must be cut and divided into the perimeter of the polygons so that the total area of the polygons are maximized and minimized. Next, we will solve the inequality problems by again computing x , the place where the wire should be severed so that the total area of these two polygons is greater than or less than a specified area, A . In addition to the above 1-cut (2-partitioned) problem, we will also analyze the above four cases for the 2-cut problem and finally generalizing it to a k -cut ($k + 1$ -partitioned problem). We will illustrate the above four cases by providing examples of specific polygons.

In our second problem, we are given two wires of specified lengths which we will bend to form two regular polygons with the total number of polygonal sides fixed. We will then calculate ways in which the total number of sides can be allocated between these two wires so that the combined area of the polygons is maximized. This will generate a non-linear coupled equation in the number of polygonal sides. This equation will be solved numerically using a computer program. Finally, we will generalize this problem to k -wires of specified lengths.

The outline of the paper is as follows: in section 2 we will examine the 2-partitioned case. Here we will devote sub-sections to deriving a general expression for A_{total} in terms of x and L , computing the exact form for x for which A_{total} is maximized and minimized, and considering lower and upper bound problems as mentioned above. Here, we also mention constraints on A and L . Finally, we illustrate each of the above scenarios with specific examples. In sections 3 and 4 we will consider an extension of section 2 examining the 3-partition and the $(k + 1)$ -partition cases, respectively. In section 5 we consider the problem where two wires of specified length are bent into two regular polygons where the total number of the polygonal sides is fixed. Here, we find how the total number of sides must be divided between the polygons so that the combined area of the two polygons is maximized. Next, we generalize these results to the case of k wires of specified lengths. Section 6 is devoted to the conclusion. In appendix A we show that the maximum area of a polygon for a fixed perimeter occurs in the limiting

case of a circle. Appendix B contains the computer program which looks for the right combination of polygonal sides that would maximize the total area. .

2 2-partitioned wire

I. The total area of two polygons

In this sub-section we will derive A_{total} , the total combined area of two regular m - and n -sided polygons. We will start by finding the area of each polygon using the formula: $Area_{polygon} = \frac{1}{2}ap$, where a and p are the apothem and perimeter of the polygon, respectively. This is equivalent to finding the area of one triangle (see accompanying figure) and multiplying it by the number of sides since the number of congruent triangles is equal to the number of sides.

Cutting length x from a total length, L , we form a regular n -gon with side length $\frac{x}{n}$. Dropping a perpendicular of length a from the center of this n -gon to a side will bisect the side. Further, the radius of the polygon will bisect the interior angle which is given by $\theta_n = (\frac{1}{2} - \frac{1}{n})\pi$. From the right triangle AOB we find $a = \frac{x}{2n} \tan \theta_n$. Hence, the area of the n -sided polygon is given by $A_n(x) = \frac{x^2}{4n} \tan \theta_n$. Since the remaining length is $y \equiv L - x$, the area of the m -sided polygon will be given by $A_m(y) = \frac{y^2}{4m} \tan \theta_m$ where $\theta_m = (\frac{1}{2} - \frac{1}{m})\pi$. Finally, the total area of the two polygons is given by the expression:

$$A_{total}(x, y) = A_n(x) + A_m(y) = \frac{x^2}{4n} \tan \theta_n + \frac{(L - x)^2}{4m} \tan \theta_m \quad (1)$$

In the case of identical polygons ($m = n$ and therefore $\theta_m = \theta_n$), Eq.1 reduces to

$$A_{total}^{(m=n)} = \frac{2x^2 - 2xL + L^2}{4n} \tan \theta_n \quad (2)$$

II. Minimum total area of two polygons

In this sub-section we consider the minima problem. We begin by finding the critical values of A_{total} . From Eq.(1) we obtain,

$$\frac{dA_{total}}{dx} = \frac{x \tan \theta_n}{2n} - \frac{(L - x) \tan \theta_m}{2m} = 0 \quad (3)$$

which results in a minimum point:

$$x_{min} = \frac{nL \tan \theta_m}{m \tan \theta_n + n \tan \theta_m} \quad (4)$$

because,

$$\frac{d^2 A_{total}}{dx^2} = \frac{\tan \theta_n}{2n} + \frac{\tan \theta_m}{2m} > 0 \quad (5)$$

This is true since, $\pi/6 \leq \theta_n, \theta_m \leq \pi/2$. Further,

$$y_{min} = \frac{mL \tan \theta_n}{m \tan \theta_n + n \tan \theta_m} \quad (6)$$

With this result of x_{min} and y_{min} , we now find expressions for $A_n(x_{min})$, $A_m(y_{min})$, and the total minimum area of the two polygons, $A_{min-total}$:

$$\begin{aligned} A_n(x_{min}) &= \frac{L^2 n \tan \theta_n \tan^2 \theta_m}{4(m \tan \theta_n + n \tan \theta_m)^2} \\ A_m(y_{min}) &= \frac{L^2 m \tan^2 \theta_n \tan \theta_m}{4(m \tan \theta_n + n \tan \theta_m)^2} \\ A_{min-total} \equiv A_{total}(x_{min}, y_{min}) &= \frac{L^2 \tan \theta_n \tan \theta_m}{4(m \tan \theta_n + n \tan \theta_m)} \end{aligned} \quad (7)$$

Further, in the case when $m = n$, we get:

$$\begin{aligned} x_{min}^{(m=n)} &= y_{min}^{(m=n)} = \frac{L}{2} \\ A_n(x_{min}) &= A_m(y_{min}) = \frac{L^2}{16n} \tan \theta_n \\ A_{min-total}^{(m=n)} &= \frac{L^2}{8n} \tan \theta_n \end{aligned} \quad (8)$$

III. Maximum total area of two polygons

This sub-section is devoted to the corresponding maxima problem. The feasible domain of x is $[0, L]$ and the only critical value is x_{min} . Therefore, using

$$\begin{aligned} A_{total}(0, L) &= \frac{L^2}{4m} \tan \theta_m \\ A_{total}(x_{min}, y_{min}) &= \frac{L^2}{4m} \tan \theta_m \left[\frac{1}{1 + \frac{n \tan \theta_m}{m \tan \theta_n}} \right] \\ A_{total}(L, 0) &= \frac{L^2}{4n} \tan \theta_n \end{aligned} \quad (9)$$

we can conclude that maximum area occurs when $x = 0$ or $x = L$ (endpoint maxima), depending on the integers m and n . This is equivalent to saying all the wire is used for the m -sided polygon (if $m > n$) or n -sided polygon (if $n > m$). Thus,

$$\begin{aligned} A_{max-total}^{(m>n)} &= \frac{L^2}{4m} \tan \theta_m \\ A_{max-total}^{(n>m)} &= \frac{L^2}{4n} \tan \theta_n \end{aligned} \quad (10)$$

IV. Lower bound on the area of two polygons

In this sub-section we will use Eq.(1) to find the possible range of values of x for which the combined area of the two polygons exceeds a given area, A . This translates to the following problem:

$$\frac{x^2}{4n} \tan \theta_n + \frac{(L-x)^2}{4m} \tan \theta_m > A \quad (11)$$

which simplifies to

$$(x - [x_{min} - \hat{x}]) (x - [x_{min} + \hat{x}]) > 0 \quad (12)$$

where we have defined for brevity

$$\hat{x} = \sqrt{\frac{mx_{min}}{nL \tan \theta_m} (4nA - Lx_{min} \tan \theta_n)} \quad (13)$$

and x_{min} is defined as before. Therefore,

$$x \in (0, x_{min} - \hat{x}) \cup (x_{min} + \hat{x}, L) \quad (14)$$

Constraints:

Firstly, \hat{x} must be real. This implies

$$L^2 - 4A \left(\frac{n}{\tan \theta_n} + \frac{m}{\tan \theta_m} \right) \leq 0 \quad (15)$$

Thus,

$$0 < L \leq L_1 \quad (16)$$

where L_1 is defined by

$$L_1 = 2\sqrt{A \left(\frac{n}{\tan \theta_n} + \frac{m}{\tan \theta_m} \right)} \quad (17)$$

Secondly, x must be non-negative, i.e.,

$$L^2 - \frac{4mA}{\tan \theta_m} \geq 0 \quad (18)$$

The solution to this inequality is

$$L \geq L_2 \quad (19)$$

with L_2 given by

$$L_2 = 2\sqrt{\frac{mA}{\tan \theta_m}} \quad (20)$$

Finally, noting that $L_1 > L_2$ and taking the intersection of the inequalities (16) and (19) we find that L must satisfy,

$$L_2 \leq L \leq L_1 \quad (21)$$

This is equivalent to

$$A_2 \leq A \leq A_1 \quad (22)$$

where

$$\begin{aligned} A_1 &= \frac{L^2 \tan \theta_m}{4m} \equiv A_{max-total}^{(m>n)} \\ A_2 &= \frac{L^2 \tan \theta_m \tan \theta_n}{4(m \tan \theta_n + n \tan \theta_m)} \equiv A_{min-total} \end{aligned} \quad (23)$$

For completeness, we state the main results of this section (Eqs. (14), (21), (22)) for the case identical polygons:

$$x \in \left(0, \frac{1}{2} \left(L - \sqrt{\frac{8nA - L^2 \tan \theta_n}{\tan \theta_n}}\right)\right) \cup \left(\frac{1}{2} \left(L + \sqrt{\frac{8nA - L^2 \tan \theta_n}{\tan \theta_n}}\right), L\right) \quad (24)$$

subject to the constraint

$$2\sqrt{\frac{nA}{\tan \theta_n}} \leq L \leq 2\sqrt{\frac{2nA}{\tan \theta_n}} \quad (25)$$

or equivalently

$$\begin{aligned} \frac{L^2 \tan \theta_n}{8n} &\leq A \leq \frac{L^2 \tan \theta_n}{4n} \\ \text{i.e., } A_{min-total}^{(m=n)} &\leq A \leq A_{max-total}^{(m=n)} \end{aligned} \quad (26)$$

V. Upper bound on the area of two polygons

Now, the problem at hand is as follows:

$$\frac{x^2}{4n} \tan \theta_n + \frac{(L-x)^2}{4m} \tan \theta_m < A \quad (27)$$

Clearly, the solution to this inequality is

$$x \in (x_{min} - \hat{x}, x_{min} + \hat{x}) \quad (28)$$

with the same constraint as before (see Eqs. 21 and 22)

VI. Examples

In this sub-section we will work out examples that will illustrate the results established in the previous sub-sections.

(a.) Minimum area problem

(α .) *hexagon* ($n = 6$, $\theta_n = \frac{\pi}{3}$) and *square* ($m = 4$, $\theta_m = \frac{\pi}{4}$)

$$\begin{aligned} x_{min} &= (2\sqrt{3} - 3)L \\ A_{min-total} &= \frac{(2 - \sqrt{3})L^2}{8} \end{aligned} \quad (29)$$

(β .) *circle* ($n \rightarrow \infty$, $\theta_n = \frac{\pi}{2}$ (see appendix A)) and *square* ($m = 4$, $\theta_m = \frac{\pi}{4}$)

$$\begin{aligned} x_{min} &= \frac{\pi L}{\pi + 4} \\ A_{min-total} &= \frac{L^2}{4(\pi + 4)} \end{aligned} \quad (30)$$

(b.) Maximum area problem

(α .) *hexagon* ($n = 6$, $\theta_n = \frac{\pi}{3}$) and *square* ($m = 4$, $\theta_m = \frac{\pi}{4}$)

$$\begin{aligned} x_{max} &= L \\ A_{max-total} &= \frac{L^2\sqrt{3}}{24} \end{aligned} \quad (31)$$

(β .) *pentagon* and *pentagon* ($m = n = 5$, $\theta_m = \theta_n = \frac{3\pi}{10}$)

$$\begin{aligned} x_{max} &= 0, L \\ A_{max-total} &= \frac{L^2\sqrt{5(5 + 2\sqrt{5})}}{100} \end{aligned} \quad (32)$$

$$\text{Note: } \tan \frac{3\pi}{10} = \sqrt{\frac{5+2\sqrt{5}}{5}}$$

(c.) Lower bound problem

(α .) *hexagon* ($n = 6$, $\theta_n = \frac{\pi}{3}$) and *square* ($m = 4$, $\theta_m = \frac{\pi}{4}$)

$$\begin{aligned} x \in \left(0, \frac{3L - \sqrt{6[8(3 + 2\sqrt{3})A - L^2\sqrt{3}]}}{3 + 2\sqrt{3}} \right) \cup \left(\frac{3L + \sqrt{6[8(3 + 2\sqrt{3})A - L^2\sqrt{3}]}}{3 + 2\sqrt{3}}, L \right) \\ 4\sqrt{A} \leq L \leq 2\sqrt{2A(2 + \sqrt{3})} \\ \frac{(2 - \sqrt{3})L^2}{8} \leq A \leq \frac{L^2}{16} \end{aligned} \quad (33)$$

(β .) *dodecagon* ($n = 12, \theta_n = \frac{5\pi}{12}$) and *triangle* ($m = 3, \theta_m = \frac{\pi}{6}$)

$$x \in \left(0, \frac{4L - 2\sqrt{12(6 + 7\sqrt{3})A - L^2(3 + 2\sqrt{3})}}{7 + 2\sqrt{3}}\right) \cup \left(\frac{4L + 2\sqrt{12(6 + 7\sqrt{3})A - L^2(3 + 2\sqrt{3})}}{7 + 2\sqrt{3}}, L\right)$$

$$2\sqrt{3A\sqrt{3}} \leq L \leq 2\sqrt{3A(8 - 3\sqrt{3})}$$

$$\frac{L^2(8 + 3\sqrt{3})}{444} \leq A \leq \frac{L^2\sqrt{3}}{36} \quad (34)$$

Note: $\tan \frac{5\pi}{12} = 2 + \sqrt{3}$

(d.) Upper bound problem

(α .) *hexagon* ($n = 6, \theta_n = \frac{\pi}{3}$) and *square* ($m = 4, \theta_m = \frac{\pi}{4}$)

$$x \in \left(\frac{3L - \sqrt{6[8(3 + 2\sqrt{3})A - L^2\sqrt{3}]}}{3 + 2\sqrt{3}}, \frac{3L + \sqrt{6[8(3 + 2\sqrt{3})A - L^2\sqrt{3}]}}{3 + 2\sqrt{3}}\right)$$

$$4\sqrt{A} \leq L \leq 2\sqrt{2A(2 + \sqrt{3})}$$

$$\frac{(2 - \sqrt{3})L^2}{8} \leq A \leq \frac{L^2}{16} \quad (35)$$

(β .) *circle* ($n \rightarrow \infty, \theta_n = \frac{\pi}{2}$) and *octagon* ($m = 8, \theta_m = \frac{3\pi}{8}$)

$$x \in \left(\frac{8L - \sqrt{8\pi[4(8 + \pi(1 + \sqrt{2}))A - L^2]}}{8 + \pi(1 + \sqrt{2})}, \frac{8L + \sqrt{8\pi[4(8 + \pi(1 + \sqrt{2}))A - L^2]}}{8 + \pi(1 + \sqrt{2})}\right)$$

$$2\sqrt{\pi A} \leq L \leq 2\sqrt{A[8(\sqrt{2} - 1) + \pi]}$$

$$\frac{L^2}{4[8(\sqrt{2} - 1) + \pi]} \leq A \leq \frac{L^2}{4\pi} \quad (36)$$

Note: $\tan \frac{3\pi}{8} = 1 + \sqrt{2}$

(e.) An explicit numerical example

Let the length of the wire be 12 units ($= L$). We would like to analyze the above four problems in the case of a square ($n = 4$) and a triangle ($m = 3$).

Firstly, to minimize the area of the polygons, the wire should be severed at $x_{min} = \frac{48}{4+3\sqrt{3}} \approx 5.220$ units so that the minimum total area of the polygons is $A_{min-total} = \frac{36}{4+3\sqrt{3}} \approx 3.915$ square units. This area is allocated between the square and triangle as $A_4|_{x=5.220} = \frac{144}{43+24\sqrt{3}} \approx 1.703$ square units and $A_3|_{x=5.220} = \frac{108\sqrt{3}}{43+24\sqrt{3}} \approx 2.212$ square units.

Secondly, the total maximum value of the area of the polygons is $A_{max-total} =$

9 square units, which means that all of the wire is used in the perimeter of the square.

For the inequality problems we first find the constraint: $\frac{36}{4+3\sqrt{3}} \leq A \leq 4\sqrt{3}$.

We choose $A = 5$ square units. Then, $x_{min} \pm \hat{x} = \frac{48 \pm 4\sqrt{3(45-16\sqrt{3})}}{4+3\sqrt{3}} \approx 8.352, 2.087$ units. Thus, in the case where we want the total area of the two polygons to exceed 5 square units the wire can be cut either in the open interval $(0, 2.087)$ or $(8.352, 12)$ units. It is interesting to note that the corresponding areas of the square and triangle vary as $A_4 \in (0, 0.272) \cup (4.360, 5)$ square units and $A_3 \in (5, 4.728) \cup (0.640, 0)$ square units. For the upper bound problem, the cutting range lies in the bounded open interval $(2.087, 8.352)$ units.

3 3-partitioned wire

I. The total area of three polygons

In this sub-section we consider the extrema problem when the wire is cut at two places; the first of length, x , and the second piece of length, y . The third segment will obviously be of length $L - x - y \equiv z$. The total area of the n -, m - and p -sided polygons is

$$A_{total}(x, y, z) = A_n(x) + A_m(y) + A_p(z) = \frac{x^2}{4n} \tan \theta_n + \frac{y^2}{4m} \tan \theta_m + \frac{z^2}{4p} \tan \theta_p \quad (37)$$

In the case of identical polygons,

$$A_{total}^{(m=n=p)} = \frac{2x^2 + 2y^2 - 2L(x+y) + 2xy + L^2}{4n} \tan \theta_n \quad (38)$$

II. Minimum total area of three polygons

For critical values,

$$\begin{aligned} \frac{\partial A_{total}}{\partial x} &= x \left(\frac{\tan \theta_n}{2n} + \frac{\tan \theta_p}{2p} \right) + y \frac{\tan \theta_p}{2p} - L \frac{\tan \theta_p}{2p} = 0 \\ \frac{\partial A_{total}}{\partial y} &= x \frac{\tan \theta_p}{2p} + y \left(\frac{\tan \theta_m}{2m} + \frac{\tan \theta_p}{2p} \right) - L \frac{\tan \theta_p}{2p} = 0 \end{aligned} \quad (39)$$

Solving these two simultaneous linear equations for x and y results in the minimum value for A_{total} :

$$x_{min} = \frac{nL \tan \theta_m \tan \theta_p}{n \tan \theta_m \tan \theta_p + m \tan \theta_n \tan \theta_p + p \tan \theta_m \tan \theta_n} \quad (40)$$

$$y_{min} = \frac{mL \tan \theta_n \tan \theta_p}{n \tan \theta_m \tan \theta_p + m \tan \theta_n \tan \theta_p + p \tan \theta_m \tan \theta_n} \quad (41)$$

because

$$\begin{aligned} \frac{\partial^2 A_{total}}{\partial x^2} \frac{\partial^2 A_{total}}{\partial y^2} - \left(\frac{\partial^2 A_{total}}{\partial x \partial y} \right)^2 &= \frac{n \tan \theta_m \tan \theta_p + m \tan \theta_n \tan \theta_p + p \tan \theta_m \tan \theta_n}{4nmp} > 0 \\ \text{and} \quad \frac{\partial^2 A_{total}}{\partial x^2} + \frac{\partial^2 A_{total}}{\partial y^2} &= \frac{\tan \theta_n}{2n} + \frac{\tan \theta_m}{2m} + \frac{\tan \theta_p}{p} > 0 \end{aligned} \quad (42)$$

Further,

$$z_{min} = \frac{pL \tan \theta_n \tan \theta_m}{n \tan \theta_m \tan \theta_p + m \tan \theta_n \tan \theta_p + p \tan \theta_m \tan \theta_n} \quad (43)$$

Therefore,

$$\begin{aligned} A_n(x_{min}) &= \frac{L^2 n \tan^2 \theta_m \tan^2 \theta_p \tan \theta_n}{4 \left(n \tan \theta_m \tan \theta_p + m \tan \theta_n \tan \theta_p + p \tan \theta_m \tan \theta_n \right)^2} \\ A_m(y_{min}) &= \frac{L^2 m \tan^2 \theta_n \tan^2 \theta_p \tan \theta_m}{4 \left(n \tan \theta_m \tan \theta_p + m \tan \theta_n \tan \theta_p + p \tan \theta_m \tan \theta_n \right)^2} \\ A_p(z_{min}) &= \frac{L^2 p \tan^2 \theta_n \tan^2 \theta_m \tan \theta_p}{4 \left(n \tan \theta_m \tan \theta_p + m \tan \theta_n \tan \theta_p + p \tan \theta_m \tan \theta_n \right)^2} \\ A_{min-total} \equiv A_{total}(x_{min}, y_{min}, z_{min}) &= \frac{L^2 \tan \theta_n \tan \theta_m \tan \theta_p}{4 \left(n \tan \theta_m \tan \theta_p + m \tan \theta_n \tan \theta_p + p \tan \theta_m \tan \theta_n \right)} \end{aligned} \quad (44)$$

In the case of identical polygons

$$\begin{aligned} x_{min}^{(m=n=p)} &= y_{min}^{(m=n=p)} = z_{min}^{(m=n=p)} = \frac{L}{3} \\ A_n(x_{min}) &= A_m(y_{min}) = A_p(z_{min}) = \frac{L^2}{36n} \tan \theta_n \\ A_{min-total}^{(m=n=p)} &= \frac{L^2}{12n} \tan \theta_n \end{aligned} \quad (45)$$

III. Maximum total area of three polygons

To compute the maximum total area, we must consider the boundary values at $x = 0$ or $y = 0$ or $z = 0$. We begin by finding the critical values y_{cr} and z_{cr} of $A_{total}(0, y, z)$. This total area, when evaluated at these critical values, becomes a candidate for the maximum total area. The results are

$$A_{total}(0, y_{cr}, z_{cr}) = \frac{L^2 \tan \theta_p \tan \theta_m}{4(p \tan \theta_m + m \tan \theta_p)}$$

$$\begin{aligned}
y_{cr} &= \frac{mL \tan \theta_p}{m \tan \theta_p + p \tan \theta_m} \\
z_{cr} &= \frac{pL \tan \theta_m}{m \tan \theta_p + p \tan \theta_m}
\end{aligned} \tag{46}$$

Similarly,

$$\begin{aligned}
A_{total}(x_{cr}, 0, z'_{cr}) &= \frac{L^2 \tan \theta_n \tan \theta_p}{4(n \tan \theta_p + p \tan \theta_n)} \\
x_{cr} &= \frac{nL \tan \theta_p}{n \tan \theta_p + p \tan \theta_n} \\
z'_{cr} &= \frac{pL \tan \theta_n}{n \tan \theta_p + p \tan \theta_n}
\end{aligned} \tag{47}$$

and

$$\begin{aligned}
A_{total}(x'_{cr}, y'_{cr}, 0) &= \frac{L^2 \tan \theta_n \tan \theta_m}{4(n \tan \theta_m + m \tan \theta_n)} \\
x'_{cr} &= \frac{nL \tan \theta_m}{n \tan \theta_m + m \tan \theta_n} \\
y'_{cr} &= \frac{mL \tan \theta_n}{n \tan \theta_m + m \tan \theta_n}
\end{aligned} \tag{48}$$

Finally, the maximum total area is given by

$$A_{max-total} = \max \left\{ A_{total}(0, y_{cr}, z_{cr}), A_{total}(x_{cr}, 0, z'_{cr}), A_{total}(x'_{cr}, y'_{cr}, 0) \right\} \tag{49}$$

IV. Upper and lower bound on the area of three polygons

We consider a simple inequality problem in which the first two polygons have the same perimeter x . As before, we look for values of x for which the combined area of these polygons is greater than or less than a specified area, A . Thus, for the lower bound problem, we can write

$$\frac{x^2}{4n} \tan \theta_n + \frac{x^2}{4m} \tan \theta_m + \frac{(L-2x)^2}{4p} \tan \theta_p > A \tag{50}$$

The solution to this inequality is given by

$$x \in (0, x_-) \cup (x_+, L) \tag{51}$$

where

$$\begin{aligned}
x_{\pm} &= \frac{1}{mp \tan \theta_n + np \tan \theta_m + 4mn \tan \theta_p} \times \\
&\quad \left\{ 2Lmn \tan \theta_p \pm (mnp[4A(mp \tan \theta_n + np \tan \theta_m + 4mn \tan \theta_p) \right. \\
&\quad \left. - L^2 \tan \theta_p (m \tan \theta_n + n \tan \theta_m)])^{\frac{1}{2}} \right\}
\end{aligned} \tag{52}$$

Obviously, the solution to the upper bound problem is

$$x \in (x_-, x_+) \quad (53)$$

The above solutions are valid provided that

$$\frac{L^2 \tan \theta_p (m \tan \theta_n + n \tan \theta_m)}{4(m p \tan \theta_n + n p \tan \theta_m + 4 m n \tan \theta_p)} \leq A \leq \frac{L^2 \tan \theta_p}{4p} \quad (54)$$

V. Example

A wire 10 units ($= L$) long is cut into three pieces. The first piece is bent to form a square ($n = 4$), the second to form a triangle ($m = 3$), and the third to form a circle ($p \rightarrow \infty$). We are interested in knowing where the cuts should be made so that the sum of the three areas is a minimum and also the manner of cutting that will produce a maximum total area.

For the minima problem, the length of the first piece is $x_{min} = \frac{40}{4+3\sqrt{3}+\pi} \approx 3.242$ units, the second piece is of length $y_{min} = \frac{30\sqrt{3}}{4+3\sqrt{3}+\pi} \approx 4.212$ units, and the third piece automatically comes out to be $z_{min} = \frac{10\pi}{4+3\sqrt{3}+\pi} \approx 2.546$ units. This gives a total minimum area of $A_{min-total} = \frac{25}{4+3\sqrt{3}+\pi} \approx 2.026$ square units with the assignment $A_4|_{x=3.242} = \frac{100}{(4+3\sqrt{3}+\pi)^2} \approx 0.657$ square units, $A_3|_{y=4.212} = \frac{75\sqrt{3}}{(4+3\sqrt{3}+\pi)^2} \approx 0.853$ square units and $A_\infty|_{z=2.546} = \frac{25\pi}{(4+3\sqrt{3}+\pi)^2} \approx 0.516$ square units.

The maximum total area of the polygons is $A_{max-total} = \frac{25\pi}{4+\pi} \approx 10.998$ square units and occurs at the boundary $y = 0$ with critical values $x_c = \frac{40}{4+\pi} \approx 5.601$ units and $z_c = \frac{10\pi}{4+\pi} \approx 4.399$ units.

In the case of the inequality problem, the area A must be chosen so that

$\frac{25(4+3\sqrt{3})}{3\pi\sqrt{3}+4\pi+48\sqrt{3}} \leq A \leq \frac{25}{\pi}$ square units. We choose $A=5$ square units. This gives $x_{\pm} = \frac{240\sqrt{3} \pm \sqrt{240\pi(9\pi+4\pi\sqrt{3}+99-20\sqrt{3})}}{3\pi\sqrt{3}+4\pi+48\sqrt{3}} \approx 6.332, 1.089$ units. Thus, in the case of the upper bound problem $x \in (1.089, 6.332)$ units while for the lower bound problem $x \in (0, 1.089) \cup (6.332, 10)$.

4 (k+1)-partitioned wire

I. The total area of the (k+1)-polygons

In this sub-section we consider the extrema problem where the wire is cut at k different places with lengths, x_1, x_2, \dots, x_k . The last segment being of length, $x_{k+1} = L - x_1 - x_2 - \dots - x_k$. Then, the total area of the n_1 -, n_2 -, ..., n_{k+1} -sided polygons is

$$\begin{aligned}
A_{total}(x_1, x_2, \dots, x_{k+1}) &= \sum_{i=1}^{k+1} A_{n_i}(x_i) = \frac{x_1^2}{4n_1} \tan \theta_{n_1} + \frac{x_2}{4n_2} \tan \theta_{n_2} + \dots + \frac{x_{k+1}^2}{4n_{k+1}} \tan \theta_{n_{k+1}} \\
&= \sum_{i=1}^{k+1} \frac{x_i^2}{4n_i} \tan \theta_{n_i}
\end{aligned} \tag{55}$$

In the case of identical polygons,

$$A_{total}^{(n_1=n_2=\dots=n_{k+1}=n)} = \frac{\tan \theta_n}{4n} \left[L^2 + 2 \sum_{i=1, i \neq j}^k (x_i^2 + x_i x_j - L x_i) \right] \tag{56}$$

II. Minimum total area of (k+1)-polygons

For critical values

$$\frac{\partial A_{total}}{\partial x_1} = \frac{\partial A_{total}}{\partial x_2} = \dots = \frac{\partial A_{total}}{\partial x_k} = 0 \tag{57}$$

with

$$\frac{\partial A_{total}}{\partial x_i} = x_i \left(\frac{\tan \theta_{n_i}}{2n_i} + \frac{\tan \theta_{n_{k+1}}}{2n_{k+1}} \right) + \frac{\tan \theta_{n_{k+1}}}{2n_{k+1}} \left(-L + \sum_{j=1, j \neq i}^k x_j \right); \quad (i = 1 \text{ to } k) \tag{58}$$

Solving these k simultaneous linear equations for x_1, x_2, \dots, x_k results in the minimum value for A_{total} :

$$x_{i_{min}} = L \left(\frac{n_i}{\tan \theta_{n_i}} \right) \left(\sum_{j=1}^{k+1} \frac{n_j}{\tan \theta_{n_j}} \right)^{-1} = L \left(\frac{n_i \prod_{j=1, j \neq i}^{k+1} \tan \theta_{n_j}}{\sum_{l=1}^{k+1} n_l \prod_{j=1, j \neq l}^{k+1} \tan \theta_{n_j}} \right); \quad (i = 1 \text{ to } k) \tag{59}$$

because

$$\det \begin{pmatrix} \frac{\partial^2 A_{total}}{\partial x_1^2} & \frac{\partial^2 A_{total}}{\partial x_1 \partial x_2} & \cdot & \cdot & \cdot & \cdot & \frac{\partial^2 A_{total}}{\partial x_1 \partial x_k} \\ \frac{\partial^2 A_{total}}{\partial x_2 \partial x_1} & \frac{\partial^2 A_{total}}{\partial x_2^2} & \cdot & \cdot & \cdot & \cdot & \frac{\partial^2 A_{total}}{\partial x_2 \partial x_k} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial^2 A_{total}}{\partial x_k \partial x_1} & \cdot & \cdot & \cdot & \cdot & \cdot & \frac{\partial^2 A_{total}}{\partial x_k^2} \end{pmatrix} > 0$$

and

$$\sum_{i=1}^k \frac{\partial^2 A_{total}}{\partial x_i^2} > 0 \tag{60}$$

Therefore,

$$A_{n_i}(x_{i_{min}}) = \frac{L^2}{4} \left(\frac{n_i \tan \theta_{n_i} \prod_{j=1, j \neq i}^k \tan^2 \theta_{n_j}}{\left[\sum_{l=1}^{k+1} n_l \prod_{j=1, j \neq l}^{k+1} \tan \theta_{n_j} \right]^2} \right); \quad (i = 1 \text{ to } k)$$

$$A_{min-total} = \frac{L^2}{4} \left(\sum_{j=1}^{k+1} \frac{n_j}{\tan \theta_{n_j}} \right)^{-1} = \frac{L^2}{4} \left(\frac{\prod_{j=1}^{k+1} \tan \theta_{n_j}}{\sum_{l=1}^{k+1} n_l \prod_{j=1, j \neq l}^{k+1} \tan \theta_{n_j}} \right) \quad (61)$$

In the case of identical polygons

$$x_{i_{min}}^{(n_1=n_2=\dots=n_{k+1})} = \frac{L}{k+1}; \quad (i = 1 \text{ to } k)$$

$$A_{n_i}^{(n_1=n_2=\dots=n_{k+1}=n)} \Big|_{x=x_{i_{min}}} = \frac{L^2}{4(k+1)^2 n} \tan \theta_n; \quad (i = 1 \text{ to } k)$$

$$A_{min-total}^{(n_1=n_2=\dots=n_{k+1}=n)} = \frac{L^2}{4(k+1)n} \tan \theta_n \quad (62)$$

III. Maximum total area of (k+1)-polygons

Here we need to consider the total area at the boundary values, $x_b = 0$; $b = 1$ to $k+1$:

$$A_{total}(x_1, x_2, \dots, x_b = 0, \dots, x_{k+1}) = \sum_{i=1, i \neq b}^{k+1} \frac{x_i^2}{4n_i} \tan \theta_{n_i}; \quad (b = 1 \text{ to } k+1) \quad (63)$$

The critical value, $x_{i_{cr}}$, of this A_{total} is given by

$$x_{i_{cr}} = L \left(\frac{n_i \prod_{j=1, j \neq i, b}^{k+1} \tan \theta_{n_j}}{\sum_{l=1, l \neq b}^{k+1} n_l \prod_{j=1, j \neq l, b}^{k+1} \tan \theta_{n_j}} \right); \quad (i = 1 \text{ to } k \text{ except } b) \quad (64)$$

The above A_{total} , when evaluated at these critical values, becomes a candidate for the maximum total area:

$$A_{total}(x_{1_{cr}}, x_{2_{cr}}, \dots, x_b = 0, \dots, x_{(k+1)_{cr}}) = \frac{L^2}{4} \left(\frac{\prod_{j=1, j \neq b}^{k+1} \tan \theta_{n_j}}{\sum_{l=1, l \neq b}^{k+1} n_l \prod_{j=1, j \neq l, b}^{k+1} \tan \theta_{n_j}} \right); \quad (b = 1 \text{ to } k+1) \quad (65)$$

and

$$A_{max-total} = \max_{b=1}^{k+1} \left\{ A_{total}(x_{1_{cr}}, x_{2_{cr}}, \dots, x_b = 0, \dots, x_{(k+1)_{cr}}) \right\} \quad (66)$$

IV. Upper and lower bound on the area of (k+1)-polygons

As in the 2-cut case, we consider the simple case in which the first k polygons have

the same perimeter, x . Therefore, the last one will have a perimeter of $L - kx$. Hence, for the lower bound problem

$$\frac{x^2}{4n_1} \tan \theta_{n_1} + \frac{x^2}{4n_2} \tan \theta_{n_2} + \dots + \frac{x^2}{4n_k} \tan \theta_{n_k} + \frac{(L - kx)^2}{4n_{k+1}} \tan \theta_{n_{k+1}} > A \quad (67)$$

This can be simplified to

$$\begin{aligned} x^2 \left[\sum_{l=1}^k \tan \theta_{n_l} \prod_{j=1, j \neq l}^{k+1} n_j + k^2 \tan \theta_{n_{k+1}} \prod_{j=1}^k n_j \right] - 2Lkx \tan \theta_{n_{k+1}} \prod_{j=1}^k n_j \\ + L^2 \tan \theta_{n_{k+1}} \prod_{j=1}^k n_j - 4A \prod_{j=1}^{k+1} n_j > 0 \end{aligned} \quad (68)$$

The solution is

$$x \in (0, x_-) \cup (x_+, L) \quad (69)$$

where

$$x_{\pm} = \frac{Lk \tan \theta_{n_{k+1}} \prod_{j=1}^k n_j \pm \sqrt{\prod_{j=1}^{k+1} n_j \sum_{l=1}^{k+1} (4A\alpha_l - L^2 \tan \theta_{n_{k+1}} \beta_l) \tan \theta_{n_l} \prod_{i=1, i \neq l}^{k+1} n_i}}{\sum_{l=1}^{k+1} \alpha_l \tan \theta_{n_l} \prod_{i=1, i \neq l}^{k+1} n_i} \quad (70)$$

and

$$\begin{aligned} \alpha_{k+1} &= k^2, & \beta_{k+1} &= 0 \\ \alpha_i &= \beta_i = 1, & \forall i &\neq k+1 \end{aligned} \quad (71)$$

Finally, the above solutions are valid if

$$\frac{L^2 \tan \theta_{n_{k+1}} \sum_{l=1}^k \tan \theta_{n_l} \prod_{j=1, j \neq l}^k n_j}{4 \sum_{l=1}^{k+1} \alpha_l \tan \theta_{n_l} \prod_{j=1, j \neq l}^{k+1} n_j} \leq A \leq \frac{L^2 \tan \theta_{n_{k+1}}}{4n_{k+1}} \quad (72)$$

The upper bound solution is

$$x \in (x_-, x_+) \quad (73)$$

V. Example

We wrap-up this section with one final concrete example. To that end, consider a wire of length 20 units. Further, assume that the wire is cut into six segments, with the first one bent into a triangle ($n_1 = 3$), the second into a square ($n_2 = 4$),

the third into a hexagon ($n_3 = 6$), the fourth into an octagon ($n_4 = 8$), the fifth into a dodecagon ($n_5 = 12$), and the sixth into a circle ($n_6 \rightarrow \infty$). As before, we consider the mannerism in which the wire should be cut so that the total area enclosed by the above six polygons is a minimum. Here we also deal with the corresponding maxima problem.

For the minima problem, the wire should be severed at $x_{1_{min}} = 4.654$ units, $x_{2_{min}} = 3.582$ units, $x_{3_{min}} = 3.103$ units, $x_{4_{min}} = 2.968$ units, $x_{5_{min}} = 2.880$ units, and $x_{6_{min}} = 2.814$ units giving a $A_{min-total} = 4.478$ square units with the allocation $A_3|_{x_1=4.654} = 1.042$ square units, $A_4|_{x_2=3.852} = 0.802$ square units, $A_6|_{x_3=3.103} = 0.695$ square units, $A_8|_{x_4=2.968} = 0.665$ square units, $A_{12}|_{x_5=2.880} = 0.645$ square units, $A_\infty|_{x_6=2.814} = 0.630$ square units.

The maximum total area of the polygons is calculated to be $A_{max-total} = 5.836$ square units and occurs at the boundary point $x_1 = 0$ with critical values, $x_{2_{cr}} = 4.669$ units, $x_{3_{cr}} = 4.043$ units, $x_{4_{cr}} = 3.868$ units, $x_{5_{cr}} = 3.753$ units, $x_{6_{cr}} = 3.667$ units.

For the inequality problem, we must choose A between 4.599 and 31.831 square units. Take $A = 23$ square units. Then, $x_\pm \approx 6.235, 0.609$ units. Thus, for the upper bound problem $x \in (0.609, 6.235)$ units, while for lower bound problem $x \in (0, 0.609) \cup (6.235, 20)$ units.

5 Extrema problem of the distribution of a fixed number of polygonal sides among an arbitrary number of wires

In this section we first consider two wires of specified lengths which are bent to form two regular polygons. The total number of sides of the polygons, I , is fixed. The question we answer is how the total number of polygonal sides, I , should be divided between the two wires so that the total area of the two polygons is maximized.

To solve the problem at hand, we consider two wires of lengths L_n and L_m . These are bent into n - and m -sided regular polygons with $n + m (= I)$ fixed. The area of the two polygons are given by $\frac{L_n^2}{4n} \tan\left(\frac{\pi}{2} - \frac{\pi}{n}\right)$ and $\frac{L_m^2}{4(I-n)} \tan\left(\frac{\pi}{2} - \frac{\pi}{I-n}\right)$. Thus,

$$A_{total} = \frac{L_n^2}{4n} \tan\left(\frac{\pi}{2} - \frac{\pi}{n}\right) + \frac{L_m^2}{4(I-n)} \tan\left(\frac{\pi}{2} - \frac{\pi}{I-n}\right) \quad (74)$$

Differentiating with respect to the number of polygonal sides, n , we get,

$$\begin{aligned} \frac{dA_{total}}{dn} &= \frac{L_n^2}{4n^2} \left[\frac{\pi}{n} + \frac{\pi}{n} \cot^2 \left(\frac{\pi}{n} \right) - \cot \left(\frac{\pi}{n} \right) \right] \\ &+ \frac{L_m^2}{4(I-n)^2} \left[-\frac{\pi}{I-n} - \frac{\pi}{I-n} \cot^2 \left(\frac{\pi}{I-n} \right) + \cot \left(\frac{\pi}{I-n} \right) \right] = 0 \end{aligned} \quad (75)$$

This gives a nonlinear equation in n :

$$(\alpha L_n)^2 [\alpha \cot^2 \alpha - \cot \alpha + \alpha] = (\beta L_m)^2 [\beta \cot^2 \beta - \cot \beta + \beta] \quad (76)$$

where

$$\begin{aligned} \alpha &= \frac{\pi}{n} \\ \beta &= \frac{\alpha \pi}{\alpha I - n} = \frac{\pi}{I - n} \end{aligned} \quad (77)$$

Finally, we generalize the above analysis for the case when there are k wires with lengths L_1, L_2, \dots, L_k . As before, we bent each of these wires into regular n_1 -, n_2 -, ..., n_k -gons with $n_1 + n_2 + \dots + n_k = I$. We look for values of n_1, n_2, \dots, n_k which would maximize the total area of the polygons. In this case, the total area of the polygons is given by

$$A_{total} = \frac{1}{4} \sum_{i=1}^k \frac{L_i^2}{n_i} \tan \left(\frac{\pi}{2} - \frac{\pi}{n_i} \right) \quad (78)$$

To solve the extrema problem we set

$$\frac{\partial A_{total}}{\partial n_1} = \frac{\partial A_{total}}{\partial n_2} = \dots = \frac{\partial A_{total}}{\partial n_{k-1}} = 0 \quad (79)$$

obtaining

$$\begin{aligned} &(\alpha_1 L_1)^2 [\alpha_1 \cot^2 \alpha_1 - \cot \alpha_1 + \alpha_1] \\ &= (\alpha_2 L_2)^2 [\alpha_2 \cot^2 \alpha_2 - \cot \alpha_2 + \alpha_2] \\ &= \dots\dots\dots \\ &= (\alpha_{k-1} L_{k-1})^2 [\alpha_{k-1} \cot^2 \alpha_{k-1} - \cot \alpha_{k-1} + \alpha_{k-1}] \\ &= (\gamma L_k)^2 [\gamma \cot^2 \gamma - \cot \gamma + \gamma] \end{aligned} \quad (80)$$

where

$$\alpha_i = \frac{\pi}{n_i} \quad \text{and} \quad \gamma = \frac{\pi}{I - \sum_{j=1}^{k-1} n_j} \quad (81)$$

Appendix B contains a C program that singles out the $\{n_1, n_2, \dots, n_k\}$ set for which the total area (Eq.(78)) attains its maximum. Rather than solving Eqs.(80) the program scans the entire subspace of permissible partitions $(n_1, n_2, \dots, n_k; n_1 + n_2 + \dots + n_k = I)$ while it keeps track of the largest value of A_{total} . Statements printing out each set and the corresponding A_{total} are included in the code, but have been commented out for the sake of execution speed.

6 Conclusions

We began this paper with a single cut of a piece of wire and computed the lengths of the segments formed so that the total combined area of the arbitrary sided polygons bent from these two segments is maximized and minimized. After solving this extrema problem, we went on to find the cutting range so that the combined area of these polygons is greater than or less than a user-specified area. Here we also mention the constraint on the selection of this user-specified area. For the reader we have worked out examples in exact form, leaving the answer in terms of the length of the wire. These examples consist of all kinds of combinations of polygons, however, for each of the above four cases, a hexagon and a square is examined to get a feel of how this combination carries itself out for each of the four cases.

We have extended the results of the above four cases to three partitioned wires and finally to any number of partitioned wires. However, for the inequality problem with more than one cut, we considered the simplified case where all of the polygons are assigned the same variable perimeter except the last one which is automatically determined from the total length of the wire. To illustrate to the reader the use of these results, we considered a 5-cut case (6-partitioned) of a wire of length 20 units. The segments are bent into a triangle, square, hexagon, octagon, dodecagon, and a circle. We found the following assignment for the perimeter of the polygons: 4.654, 3.582, 3.103, 2.968, 2.880, and 2.814 units respectively, so that the combined area of these polygons is minimized with this area given by 4.478 square units. For the total area of the polygons to be maximized, we found the following allocation for the perimeter of the polygons: 4.669 for the square, 4.043 for the hexagon, 3.868 for the octagon, 3.753 for the dodecagon, and 3.667 for the circle. Note that for this case none of the wire is used for the triangle. For the inequality problem, we chose the user-specified area to be 23 square units from a possible range of 4.599 to 31.831 square units. So, for the upper bound problem, this result gives a cutting

range between 0.609 and 6.235 units for the first five polygons and for the lower bound problem a cutting range between 0 and 0.609 or 6.235 and 20 units.

In section 5 we explored the case where the number of sides of the polygons formed is fixed and we calculated how this fixed number of sides can be allocated firstly, among two wires, and then an arbitrary number of wires so that the total area of the polygons is maximized.

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8 Appendix A

In this appendix we will show that the maximum area, \mathcal{A}_n , of an n -sided polygon (with perimeter, P) is attained in the limiting case of a circle: $n \rightarrow \infty, \theta_n = \pi/2$.

$$\begin{aligned} \mathcal{A}_n &= \frac{P^2}{4n} \tan \theta_n; \quad \theta_n = \left(\frac{1}{2} - \frac{1}{n}\right)\pi \\ \frac{d\mathcal{A}_n}{d\theta_n} &= \left[\frac{P(\pi - 2\theta_n)}{4\pi} \right]^2 \left[\left(\frac{\pi}{2} - \theta_n\right) \frac{1}{\cos^2 \theta_n} - \frac{\sin \theta_n}{\cos \theta_n} \right] = 0 \end{aligned} \quad (82)$$

so that

$$\begin{aligned} \text{either} \quad & \pi - 2\theta_n = 0 \\ \text{or} \quad & \sin^4 \theta_n - \sin^2 \theta_n + (\pi/2 - \theta_n)^2 = 0 \end{aligned} \quad (83)$$

Clearly, the only solution to these equations is

$$\theta_n = \frac{\pi}{2} \quad (84)$$

Therefore,

$$\begin{aligned} \frac{\pi}{2} &= \left(\frac{1}{2} - \frac{1}{n}\right)\pi \\ &\Rightarrow n \rightarrow \infty \end{aligned} \quad (85)$$

The acceptable values for n are $3 \leq n < \infty$, this implies $\pi/6 \leq \theta_n \leq \pi/2$. $d\mathcal{A}_n/d\theta_n < 0$ for θ_n in $[\pi/3, \pi/2)$ and $d\mathcal{A}_n/d\theta_n > 0$ for θ_n in $(\pi/2, \infty)$. This implies \mathcal{A}_n attains its maximum at $\theta_n = \pi/2$.

Next, we calculate the corresponding maximum area,

$$\mathcal{A}_{n-max} = \frac{P^2}{8\pi} (\pi - 2\theta_n) \tan \theta_n |_{\theta_n = \frac{\pi}{2}}$$

This is in the indeterminate form: $0 \times \infty$. We put the above expression in the form $\frac{\infty}{\infty}$ and apply L'Hopital's rule successively to carry out the limiting procedure:

$$\begin{aligned} \mathcal{A}_{n-max} &= \frac{P^2}{8\pi} \lim_{\theta_n \rightarrow \frac{\pi}{2}} \frac{\tan \theta_n}{\frac{1}{\pi - 2\theta_n}} \\ &= \frac{P^2}{16\pi} \lim_{\theta_n \rightarrow \frac{\pi}{2}} \frac{(\pi - 2\theta_n)^2}{\cos^2 \theta_n} \\ &= \frac{P^2}{4\pi} \lim_{\theta_n \rightarrow \frac{\pi}{2}} \frac{(\pi - 2\theta_n)}{\sin 2\theta_n} \\ &= -\frac{P^2}{4\pi} \lim_{\theta_n \rightarrow \frac{\pi}{2}} \frac{1}{\cos 2\theta_n} \\ &= \frac{P^2}{4\pi} \\ &= \pi r^2 \quad (P = C = 2\pi r) \end{aligned} \tag{86}$$

Obviously the minimum area occurs for the case of a triangle:

$$\begin{aligned} \mathcal{A}_{n-min} &= \frac{P^2}{4n} \tan \left[\left(\frac{1}{2} - \frac{1}{n} \right) \pi \right]_{n=3} \\ &= \frac{P^2}{12\sqrt{3}} \end{aligned} \tag{87}$$

9 Appendix B

The following is the computer code that looks for the $\{n_1, n_2, \dots, n_k\}$ set of polygonal sides which maximizes the value of total area in Eq.(78). This appendix also contains four outputs of the program.

10 References

1. Varberg, D. and Purcell, E. J., *Calculus with Analytic Geometry*. 6th ed. New Jersey:Prentice Hall, 1992.
2. Embry, M. R., Schell, J. F. and Thomas, J. P., *Calculus and Linear Algebra*. Philadelphia:W.B. Saunders Company, 1972.
3. Apostol, T. M., *Mathematical Analysis*. Reading:Addison-Wesley, 1958.